

On Gauss-Pólya's Inequality

By

J. Pečarić, J. Šunde, and S. Varošaneć

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Abstract

Let $g, b: [a, b] \rightarrow \mathbf{R}$ be nonnegative nondecreasing functions such that g and b have a continuous first derivative and $g(a) = b(a), g(b) = b(b)$. Let $p = (p_1, p_2)$ be a pair of positive real numbers p_1, p_2 such that $p_1 + p_2 = 1$.

a) If $f: [a, b] \rightarrow \mathbf{R}$ be a nonnegative nondecreasing function, then for $r, s < 1$

$$M_p^{[r]} \left(\int_a^b g'(t) f(t) dt, \int_a^b b'(t) f(t) dt \right) \leq \int_a^b (M_p^{[s]}(g(t), b(t)))' f(t) dt \quad (1)$$

holds, and for $r, s > 1$ the inequality is reversed.

b) If $f: [a, b] \rightarrow \mathbf{R}$ is a nonnegative nonincreasing function then for $r < 1 < s$ (1) holds and for $r > 1 > s$ the inequality is reversed.

Similar results are derived for quasiarithmetic and logarithmic means.

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1. Introduction

Gauss mentioned the following result in [2]:

If f is a nonnegative and decreasing function then

$$\left(\int_0^\infty x^2 f(x) dx \right)^2 \leq \frac{5}{9} \int_0^\infty f(x) dx \int_0^\infty x^4 f(x) dx. \quad (2)$$

Pólya and Szegő classical book “Problems and Theorems in Analysis, I” [7] gives the following generalization and extension of Gauss’ inequality (2).

Theorem A. (Pólya’s inequality) *Let a and b be nonnegative real numbers.*

a) If $f: [0, \infty) \rightarrow \mathbf{R}$ is a nonnegative and decreasing function, then

$$\left(\int_0^\infty x^{a+b} f(x) dx \right)^2 \leq \left(1 - \left(\frac{a-b}{a+b+1} \right)^2 \right) \int_0^\infty x^{2a} f(x) dx \times \int_0^\infty x^{2b} f(x) dx \quad (3)$$

whenever the integrals exist.

b) If $f: [0, 1) \rightarrow \mathbf{R}$ is a nonnegative and increasing function, then

$$\left(\int_0^1 x^{a+b} f(x) dx \right)^2 \geq \left(1 - \left(\frac{a-b}{a+b+1} \right)^2 \right) \int_0^1 x^{2a} f(x) dx \times \int_0^1 x^{2b} f(x) dx. \quad (4)$$

Obviously, putting $a = 0$ and $b = 2$ in (3) we obtain Gauss’ inequality. Recently Pečarić and Varošanec [6] obtained a generalization.

Theorem B. *Let $f: [a, b] \rightarrow \mathbf{R}$ be nonnegative and increasing, and let $x_i: [a, b] \rightarrow \mathbf{R}$ ($i = 1, \dots, n$) be nonnegative increasing functions with a continuous first derivative. If p_i , ($i = 1, \dots, n$) are positive real numbers such that $\sum_{i=1}^n \frac{1}{p_i} = 1$, then*

$$\int_a^b \left(\prod_{i=1}^n (x_i(t))^{1/p_i} \right)' f(t) dt \geq \prod_{i=1}^n \left(\int_a^b x_i'(t) f(t) dt \right)^{1/p_i} \quad (5)$$

If $x_i(a) = 0$ for all $i = 1, \dots, n$ and if f is a decreasing function then the reverse inequality holds.

The previous result is an extension of the Pólya’s inequality. If we substitute in (5): $n = 2, p_1 = p_2 = 2, a = 0, b = 1, g(x) = x^{2u+1}, h(x) = x^{2v+1}$ where $u, v > 0$, we have (4).

In this paper we provide generalizations of Theorem B in a number of directions. In Section 2 we first provide the inequality for weighted means. We note that, as is suggested by notation for means, our result extends to the case when the ordered pair (p_1, p_2) is replaced by an n -tuple. We derive also a version of our theorem for higher derivatives.

Section 4 treats some corresponding results when M is replaced by quasiarithmetic mean. This can be done when the function involved enjoys appropriate convexity properties. A second theorem in Section 4 allows one weight p_1 to be positive and the others negative.

Section 5 addresses the logarithmic mean.

2. Results Connected with Weighted Means

$M_p^{[s]}(a)$ denotes the weighted mean of order r and weights $p = (p_1, \dots, p_n)$ of a positive sequence $a = (a_1, \dots, a_n)$. The n -tuple p is of positive numbers p_i with $\sum_{i=1}^n p_i = 1$. The mean is defined by

$$M_p^{[r]}(a) = \begin{cases} \left(\sum_{i=1}^n p_i a_i^r \right)^{1/r} & \text{for } r \neq 0 \\ \prod_{i=1}^n a_i^{p_i} & \text{for } r = 0. \end{cases}$$

In the special cases $r = -1, 0, 1$ we obtain respectively the familiar harmonic, geometric and arithmetic mean.

The following theorem, which is a simple consequence of Jensen's inequality for convex functions, is one of the most important inequalities between means.

Theorem C. *If a and p are positive n -tuples and $s < t$, $s, t \in \mathbf{R}$, then*

$$M_p^{[s]}(a) \leq M_p^{[t]}(a) \quad \text{for } s < t, \quad (6)$$

with equality if and only if $a_1 = \dots = a_n$.

A well-known consequence of the above statement is the inequality between arithmetic and geometric means. Previous results and refinements can be found in [3].

The following theorem is the generalization of Theorem B.

Theorem 1. *Let $g, h: [a, b] \rightarrow \mathbf{R}$ be nonnegative nondecreasing functions such that g and h have a continuous first derivative and $g(a) = h(a), g(b) = h(b)$. Let $p = (p_1, p_2)$ be a pair of positive real numbers p_1, p_2 such that $p_1 + p_2 = 1$.*

a) If $f: [a, b] \rightarrow \mathbf{R}$ be a nonnegative nondecreasing function, then for $r, s < 1$

$$M_p^{[r]} \left(\int_a^b g'(t) f(t) dt, \int_a^b h'(t) f(t) dt \right) \leq \int_a^b \left(M_p^{[s]}(g(t), h(t)) \right)' f(t) dt \quad (7)$$

holds, and for $r, s > 1$ the inequality is reversed.

b) If $f: [a, b] \rightarrow \mathbf{R}$ is a nonnegative nonincreasing function then for $r < 1 < s$ (7) holds and for $r > 1 > s$ the inequality is reversed.

Proof: Let us suppose that $r, s < 1$ and f is nondecreasing. Using inequality (6) we obtain

$$\begin{aligned}
 & M_p^{[r]} \left(\int_a^b g'(t) f(t) dt, \int_a^b b'(t) f(t) dt \right) \\
 & \leq M_p^{[1]} \left(\int_a^b g'(t) f(t) dt, \int_a^b b'(t) f(t) dt \right) \\
 & = \int_a^b (p_1 g'(t) + p_2 b'(t)) f(t) dt \\
 & = f(b) M_p^{[1]}(g(b), b(b)) - f(a) M_p^{[1]}(g(a), b(a)) \\
 & \quad - \int_a^b M_p^{[1]}(g(t), b(t)) df(t) \\
 & \leq f(b) M_p^{[1]}(g(b), b(b)) - f(a) M_p^{[1]}(g(a), b(a)) \\
 & \quad - \int_a^b M_p^{[s]}(g(t), b(t)) df(t) \\
 & = f(b) M_p^{[1]}(g(b), b(b)) - f(a) M_p^{[1]}(g(a), b(a)) \\
 & \quad - \left(f(b) M_p^{[s]}(g(b), b(b)) - f(a) M_p^{[s]}(g(a), b(a)) \right. \\
 & \quad \left. - \int_a^b (M_p^{[s]}(g(t), b(t)))' f(t) dt \right) \\
 & = f(b) \left(M_p^{[1]}(g(b), b(b)) - M_p^{[s]}(g(b), b(b)) \right) \\
 & \quad - f(a) \left(M_p^{[1]}(g(a), b(a)) - M_p^{[s]}(g(a), b(a)) \right) \\
 & \quad + \int_a^b \left(M_p^{[s]}(g(t), b(t)) \right)' f(t) dt \\
 & = \int_a^b \left(M_p^{[s]}(g(t), b(t)) \right)' f(t) dt.
 \end{aligned}$$

A similar proof applies in each of the other cases. \square

Remark 1. In Theorem 1 we deal with two functions g and b . Obviously a similar result holds for n functions x_1, \dots, x_n which satisfy the same conditions as g and b .

Remark 2. It is obvious that on substituting $r = s = 0$ into (7) we have inequality (5) for $n = 2$. The result for $r = s = 0$ is given in [1].

In the following theorem we consider an inequality involving higher derivatives.

Theorem 2. Let $f: [a, b] \rightarrow \mathbf{R}$, $x_i: [a, b] \rightarrow \mathbf{R}$ ($i = 1, \dots, m$) be nonnegative functions with continuous n -th derivatives such that $x_i^{(n)}$, ($i = 1, \dots, m$) are nonnegative functions and p_i , ($i = 1, \dots, m$) be positive real numbers such that $\sum_{i=1}^m p_i = 1$.

a) If $(-1)^{n-1} f^{(n)}$ is a nonnegative function, then for $r, s < 1$

$$\begin{aligned} & M_p^{[r]} \left(\int_a^b x_1^{(n)}(t) f(t) dt, \dots, \int_a^b x_m^{(n)}(t) f(t) dt \right) \\ & \leq \Delta + \int_a^b \left(M_p^{[s]}(x_1(t), \dots, x_m(t)) \right)^{(n)} f(t) dt \end{aligned} \quad (8)$$

holds, where

$$\begin{aligned} \Delta = & \sum_{k=0}^{n-1} (-1)^{n-k-1} f^{(n-k-1)}(t) \\ & \left(\sum_{i=1}^m p_i x_i^{(k)}(t) - \left(M_p^{[s]}(x_1(t), \dots, x_m(t)) \right)^{(k)} \right) \Bigg|_a^b \end{aligned}$$

If

$$x_i^{(k)}(a) = x_i^{(k)}(a) \text{ and } x_i^{(k)}(b) = x_i^{(k)}(b) \text{ for } i, j \in \{1, \dots, m\} \quad (9)$$

and $k = 0, \dots, n-1$, then

$$\begin{aligned} & M_p^{[r]} \left(\int_a^b x_1^{(n)}(t) f(t) dt, \dots, \int_a^b x_m^{(n)}(t) f(t) dt \right) \\ & \leq \int_a^b \left(M_p^{[s]}(x_1(t), \dots, x_m(t)) \right)^{(n)} f(t) dt. \end{aligned} \quad (10)$$

If $r, s > 1$, then the inequalities (8) and (10) are reversed.

b) If $(-1)^n f^{(n)}$ is a nonnegative function, then for $r < 1 < s$ the inequalities (8) and (10) hold and for $r > 1 > s$ they are reversed.

Proof: a) Let r and s be less than 1. Integrating by part n -times and using (6), we obtain

$$\begin{aligned}
 & M_p^{[r]} \left(\int_a^b x_1^{(n)}(t) f(t) dt, \quad \int_a^b x_m^{(n)}(t) f(t) dt \right) \\
 & \leq M_p^{[1]} \left(\int_a^b x_1^{(n)}(t) f(t) dt, \dots, \int_a^b x_m^{(n)}(t) f(t) dt \right) \\
 & = \left(\sum_{k=0}^{n-1} (-1)^{n-k-1} f^{(n-k-1)}(t) \sum_{i=1}^m p_i x_i^{(k)}(t) \right) \Big|_a^b \\
 & \quad - \int_a^b M_p^{[1]}(x_1(t), \dots, x_m(t)) (-1)^{(n-1)} f^{(n)}(t) dt \\
 & \leq \left(\sum_{k=0}^{n-1} (-1)^{n-k-1} f^{(n-k-1)}(t) \sum_{i=1}^m p_i x_i^{(k)}(t) \right) \Big|_a^b \\
 & \quad - \int_a^b M_p^{[s]}(x_1(t), \dots, x_m(t)) (-1)^{(n-1)} f^{(n)}(t) dt \\
 & = \Delta + \int_a^b \left(M_p^{[s]}(x_1(t), \dots, x_m(t)) \right)^{(n)} f(t) dt.
 \end{aligned}$$

We shall prove that $\Delta = 0$ if $x_i, i = 1, \dots, m$, satisfy (9).

Let us use notation $\mathcal{A}_k = x_i^{(k)}(a)$ for $k = 0, 1, \dots, n-1$. Then $\sum_{i=1}^m p_i x_i^{(k)}(a) = \mathcal{A}_k$. Consider the k -th order derivative of function $y^{(p)}$ where y is an arbitrary function with k -th order derivative. First, there exists function $\phi_k^{[p]}$ such that

$$(y^{(p)})^{(k)} = \phi_k^{[p]}(y, y', \dots, y^{(k)}).$$

This follows by induction on k . For $k = 1$ we have $(y^{(p)})' = p y^{p-1} y' = \phi_1^{[p]}(y, y')$. Suppose that proposition is valid for all $j < k+1$. Then using Leibniz's rule we get

$$\begin{aligned}
 (y^{(p)})^{(k+1)} &= (p y^{p-1} y')^{(k)} \\
 &= p \sum_{j=0}^k \binom{k}{j} (y^{p-1})^{(j)} (y')^{(k-j)} \\
 &= p \sum_{j=0}^k \binom{k}{j} \phi_j^{[p-1]}(y, y', \dots, y^{(j)}) y^{(k-j+1)} \\
 &= \phi_{k+1}^{[p]}(y, y', \dots, y^{(k+1)}).
 \end{aligned} \tag{11}$$

Suppose that $s \neq 0$ and use the abbreviated notation $M(t)$ for the mean $M_p^{[s]}(x_1(t), \dots, x_m(t))$. Then $M^s(t) = \sum_{i=1}^m P_i x_i^s(t)$. The statement " $M^{(k)}(a) = \mathcal{A}_k$ " will be proved by induction on k . It is easy to check for $k = 0$ and $k = 1$.

Suppose it holds for all $j < k + 1$. Then

$$\begin{aligned} \left(\sum_{i=1}^m p_i x_i^s(t) \right)^{(k+1)} \Big|_{t=a} &= \sum_{i=1}^m p_i \phi_{(k+1)}^{[s]}(x_i(t), x_i'(t), \dots, x_i^{(k+1)}(t)) \Big|_{t=a} \\ &= \phi_{(k+1)}^{[s]}(\mathcal{A}_0, \mathcal{A}_1, \dots, \mathcal{A}_{k+1}) \\ &= {}^s \sum_{j=0}^k \binom{k}{j} \phi_j^{[s-1]}(\mathcal{A}_0, \mathcal{A}_1, \dots, \mathcal{A}_j) \mathcal{A}_{k-j+1} \\ &\quad + \phi_k^{[s-1]}(\mathcal{A}_0, \mathcal{A}_1, \dots, \mathcal{A}_k) \mathcal{A}_{k+1}. \end{aligned}$$

On the other hand, using (11) we get

$$\begin{aligned} (M^s(t))^{(k+1)} \Big|_{t=a} &= {}^s \sum_{j=0}^k \binom{k}{j} \phi_j^{[s-1]}(M(a), M'(a), \dots, M^{(j)}(a)) \\ &\quad \times M^{(k-j+1)}(a) + \phi_k^{[s-1]}(M(a), M'(a), \dots, M^{(k)}(a)) M^{(k+1)}(a) \\ &= {}^s \sum_{j=0}^k \binom{k}{j} \phi_j^{[s-1]}(\mathcal{A}_0, \mathcal{A}_1, \dots, \mathcal{A}_j) \mathcal{A}_{k-j+1} + \phi_k^{[s-1]} \\ &\quad (\mathcal{A}_0, \mathcal{A}_1, \dots, \mathcal{A}_k) M^{(k+1)}(a). \end{aligned}$$

Comparing these two results we obtain that $M^{(k+1)}(a) = \mathcal{A}_{k+1}$, which is enough to conclude that $\Delta = 0$.

In the other cases the proof is similar, except in the case $s = 0$ which is left to the reader. \square

3. Applications

Now we will restrict our attention to the case when $r = 0$ and the x_i are power functions.

The case when $n = 1$.

Set: $r = 0, n = 1, a = 0, b = 1, x_i(t) = t^{a_i p_i + 1}$ in (8), where $a_i > -\frac{1}{p_i}$ for $i = 1, \dots, m, p_i > 0$ and $\sum_{i=1}^m \frac{1}{p_i} = 1$. We obtain that $\Delta = 0$ and

$$\int_0^1 t^{a_1 + \dots + a_m} f(t) dt \geq \frac{\prod_{i=1}^m (a_i p_i + 1)^{1/p_i}}{1 + \sum_{i=1}^m a_i} \prod_{i=1}^m \left(\int_0^1 t^{a_i p_i} f(t) dt \right)^{1/p_i} \quad (12)$$

if f is a nondecreasing function. It is an improvement of Pólya's inequality (4). Some other results related to this inequality can be found in [5] and [8].

For example, combining (12) and the inequality

$$\sum_{i=1}^m a_i + 2 \geq \prod_{i=1}^m (a_i p_i + 2)^{1/p_i}$$

which follows from the inequality between arithmetic and geometric means, we obtain

$$\begin{aligned} \int_0^1 t^{a_1 + \dots + a_m} f(t) dt &\geq \frac{\prod_{i=1}^m ((a_i p_i + 1)(a_i p_i + 2))^{1/p_i}}{(1 + \sum_{i=1}^m a_i)(2 + \sum_{i=1}^m a_i)} \\ &\times \prod_{i=1}^m \left(\int_0^1 t^{a_i p_i} f(t) dt \right)^{1/p_i} \end{aligned} \quad (13)$$

The case when $n = 2$.

Set: $r = 0, n = 2, a = 0, b = 1, x_i(t) = t^{a_i p_i + 2}$ in (8), where $a_i > -\frac{1}{p_i}$ for $i = 1, \dots, m, p_i > 0$ and $\sum_{i=1}^m \frac{1}{p_i} = 1$. After some simple calculation, we obtain that $\Delta = 0$ and inequality (13) holds if f is a concave function. So inequality (13) applies not only for f nondecreasing, but also for f concave.

4. Results for Quasiarithmetic Means

Definition 2. Let f be a monotone real function with inverse f^{-1} , $p = (p_1, \dots, p_n) = (p_i)_i$, $a = (a_1, \dots, a_n) = (a_i)_i$ be real n -tuples. The quasiarithmetic mean of n -tuple a is defined by

$$M_f(a; p) = f^{-1} \left(\frac{1}{P_n} \sum_{i=1}^n p_i f(a_i) \right),$$

where $P_n = \sum_{i=1}^n p_i$.

For $p_i \geq 0, P_n = 1, f(x) = x^r (r \neq 0)$ and $f(x) = \ln x (r = 0)$ the quasiarithmetic mean $M_f(a; p)$ is the weighted mean $M_p^{[r]}(a)$ of order r .

Theorem 3. Let p be a positive n -tuple, $x_i: [a, b] \rightarrow \mathbf{R} (i = 1, \dots, n)$ be non-negative functions with continuous first derivative such that $x_i(a) = x_j(a), x_i(b) = x_j(b), i, j = 1, \dots, n$

a) If φ is a nonnegative nondecreasing function on $[a, b]$ and if f and g are convex increasing or concave decreasing functions, then

$$M_f \left(\left(\int_a^b x'_i(t) \varphi(t) dt \right); p \right) \geq \int_a^b M'_g((x_i(t))_i; p) \varphi(t) dt. \quad (14)$$

If f and g are concave increasing or convex decreasing functions, the inequality is reversed.

b) If φ is a nonnegative nonincreasing function on $[a, b]$, f convex increasing or concave decreasing function and g is concave increasing or convex decreasing, then (14) holds.

If f is concave increasing or convex decreasing function and g is convex increasing or concave decreasing, then (14) is reversed.

Proof: Suppose that φ is nondecreasing and f and g are convex functions. We shall use integration by parts and the well-known Jensen inequality for convex functions. The latter states that if (p_i) is a positive n -tuple and $a_i \in I$, then for every convex function $f: I \rightarrow \mathbb{R}$ we have

$$f\left(\frac{1}{P_n} \sum_{i=1}^n p_i a_i\right) \leq \frac{1}{P_n} \sum_{i=1}^n p_i f(a_i). \quad (15)$$

We have

$$\begin{aligned} M_f\left(\left(\int_a^b x'_i(t)\varphi(t) dt\right)_i; p\right) &= f^{-1}\left(\frac{1}{P_n} \sum_{i=1}^n p_i f\left(\int_a^b x_i(t)\varphi(t) dt\right)\right) \\ &\geq \frac{1}{P_n} \sum_{i=1}^n p_i \int_a^b x'_i(t)\varphi(t) dt = \int_a^b \frac{1}{P_n} \left(\sum_{i=1}^n p_i x'_i(t)\right) \varphi(t) dt \\ &= \frac{1}{P_n} \sum_{i=1}^n p_i x_i(t)\varphi(t)\Big|_a^b - \int_a^b \frac{1}{P_n} \left(\sum_{i=1}^n p_i x_i(t)\right) d\varphi(t) \\ &\geq \frac{1}{P_n} \sum_{i=1}^n p_i x_i(t)\varphi(t)\Big|_a^b - \int_a^b g^{-1}\left(\frac{1}{P_n} \left(\sum_{i=1}^n p_i g(x_i(t))\right)\right) d\varphi(t) \\ &= \frac{1}{P_n} \sum_{i=1}^n p_i x_i(t)\varphi(t)\Big|_a^b - \int_a^b M_g(x_i(t)); p) d\varphi(t) \\ &= \frac{1}{P_n} \sum_{i=1}^n p_i x_i(t)\varphi(t)\Big|_a^b - M_g((x_i(t)); p)\varphi(t)\Big|_a^b \\ &\quad + \int_a^b M'_g((x_i(t)); p)\varphi(t) dt = \int_a^b M'_g((x_i(t)); p)\varphi(t) dt. \quad \square \end{aligned}$$

Theorem 4. Let $x_i, i = 1, \dots, n$, satisfy assumptions of Theorem 4 and let p be a real n -tuple such that

$$p_1 > 0, \quad p_i \leq 0 \quad (i = 2, \dots, n), \quad P_n > 0. \quad (16)$$

a) If φ is a nonnegative nonincreasing function on $[a, b]$ and if f and g are concave increasing or convex decreasing functions, then (14) holds, while if f and g are convex increasing or concave decreasing (14) is reversed.

b) If φ is a nonnegative nondecreasing function on $[a, b]$, f is convex increasing or concave decreasing and g concave increasing or convex decreasing, then (14) holds.

If f is concave increasing or convex decreasing and g is convex increasing or concave decreasing, then (14) is reversed.

The proof is similar to that of Theorem 4. Instead of Jensen's inequality, a reverse Jensen's inequality [3, p. 6] is used: that is, if p_i is real n -tuple such that (16) holds, $a_i \in I, i = 1, \dots, n$, and $(1/P_n) \sum_{i=1}^n p_i a_i \in I$, then for every convex function $f: I \rightarrow \mathbb{R}$ (15) is reversed.

Remark 3. In Theorem 4 and 5 we deal with first derivatives. We can state an analogous result for higher-order derivatives as in Section 2.

Remark 4. The assumption that p is a positive n -tuple in Theorem 4 can be weakened to p being a real n -tuple such that

$$0 \leq \sum_{i=1}^k p_i \leq P_n \quad (1 \leq k \leq n), \quad P_n > 0$$

and $(\int x'_i(t)\varphi(t) dt)_i$ and $(x_i(t))_i, t \in [a, b]$ being monotone n -tuples.

In that case, we use Jensen-Steffensen's inequality [3, p. 6]. instead of Jensen's in-equality in the proof.

In Theorem 5, the assumption on n -tuple p can be replaced by p being a real n -tuple such that for some $k \in \{1, \dots, m\}$

$$\sum_{i=1}^k p_i \leq 0 (k < m) \quad \text{and} \quad \sum_{i=k}^n p_i \leq 0 (k > m)$$

and $(\int x'_i(t)\varphi(t) dt)_i, (x_i(t))_i, t \in [a, b]$ being monotone n -tuples.

We use the reverse Jensen-Steffensen's inequality (see [3, p. 6] and [4]) in the proof.

5. Results for Logarithmic Means

We define the logarithmic mean $L_r(x, y)$ of distinct positive numbers x, y by

$$L_r(x, y) = \begin{cases} \left(\frac{1}{y-x} \frac{y^{r+1} - x^{r+1}}{r+1} \right)^{1/r} & r \neq -1, 0 \\ \frac{1}{e} \left(\frac{y^y}{x^x} \right)^{\frac{1}{y-x}} & r = 0 \\ \frac{\ln y - \ln x}{y-x} & r = -1 \end{cases}$$

and take $L_r(x, x) = x$. The function $r \mapsto L_r(x, y)$ is nondecreasing.

It is easy to see that $L_1(x, y) = \frac{x+y}{2}$ and using method similar to that of the previous theorems we obtain the following result.

Theorem 5. Let $g, b: [a, b] \mapsto \mathbf{R}$ be nonnegative nondecreasing functions with continuous first derivatives and $g(a) = b(a), g(b) = b(b)$.

a) If f is a nonnegative increasing function on $[a, b]$, and if $r, s \leq 1$, then

$$L_r \left(\int_a^b g'(t) f(t) dt, \int_a^b b'(t) f(t) dt \right) \leq \int_a^b L'_s(g(t), b(t)) f(t) dt. \quad (16)$$

If $r, s \geq 1$ then the reverse inequality holds.

b) If f is a nonnegative nonincreasing function then for $r < 1 < s$ (16) holds, and for $r > 1 > s$ the reverse inequality holds.

Proof: Let f be a nonincreasing function and $r < 1 < s$. Using $F = -f$, integration by parts and inequalities between logarithmic means we get

$$\begin{aligned} & L_r \left(\int_a^b g'(t) f(t) dt, \int_a^b b'(t) f(t) dt \right) \\ & \leq L_1 \left(\int_a^b g'(t) f(t) dt, \int_a^b b'(t) f(t) dt \right) = \frac{1}{2} \int_a^b (g(t) + b(t))' f(t) dt \\ & = \frac{1}{2} (g(t) + b(t)) f(t) \Big|_a^b + \int_a^b \frac{1}{2} (g(t) + b(t)) dF(t) \\ & \leq \frac{1}{2} (g(t) + b(t)) f(t) \Big|_a^b + \int_a^b L_s(g(t), b(t)) dF(t) \\ & = \frac{1}{2} (g(t) + b(t)) f(t) \Big|_a^b - L_s(g(t), b(t)) f(t) \Big|_a^b \\ & \quad + \int_a^b L'_s(g(t), b(t)) f(t) dt = \int_a^b L'_s(g(t), b(t)) f(t) dt. \end{aligned}$$

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Authors' addresses: J. Pečarić, Fac. of Textile Techn. University of Zagreb, Pierottijeva 6, 10000 Zagreb, Croatia; J. Šunde,¹ Def. Science and Tech. Org., Communication Division, PO Box 1500, Salisbury SA 5108, Australia; S. Varošaneć, Dept. of Mathematics, University of Zagreb, Bijenička 30, 10000 Zagreb, Croatia.

¹ This work was completed while author was at University of Adelaide.